

Controllability of Formations over Time-varying Graphs

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Abstract—In this paper, we investigate the controllability of a class of formation control systems. Given a directed graph, we assign an agent to each of its vertices and let the edges of the graph describe the information flow in the system. We relate the strongly connected components of this graph to the reachable set of the formation control system. Moreover, we show that the formation control model is approximately path-controllable over a path-connected, open dense subset as long as the graph is weakly connected and satisfies some mild assumption on the numbers of vertices of the strongly connected components.

I. INTRODUCTION

We investigate here the controllability and path-controllability of a non-linear formation control system with N agents in \mathbb{R}^n . As is usually done, we use a directed graph $G = (V, E)$, with vertex set $V = \{1, \dots, N\}$ and edge set E , to describe the information flow in the system. We denote by $i \rightarrow j$ an edge in G . Precisely, to each vertex corresponds an agent and by a slight abuse of notation, we refer to agent i as $x_i \in \mathbb{R}^n$. Denote by V_i^- the set of out-neighbors of i : $V_i^- := \{j \in V \mid i \rightarrow j \in E\}$. The motion of agent x_i is given by

$$\dot{x}_i = \sum_{j \in V_i^-} u_{ij}(t, x)(x_j - x_i) \quad (1)$$

where each u_{ij} is an integrable real-valued function. This formation control model and variations of it have been widely investigated in recent years [1]–[11]. Questions about how these scalar functions, u_{ij} 's, are designed to organize multi-agent systems [1], [2], questions about convergence of the dynamics [3], questions about local/global stabilization of the target formation [10], [11], and questions about robustness of the formation control laws [6]–[9] have all been investigated to some extent.

In this paper, we investigate whether we can steer the multi-agent system (1) from any initial configuration to any target configuration through the choice of the u_{ij} 's. The same question was addressed earlier for an undirected graph [12]. It was shown in [12] that if the undirected graph G is connected and $(N - n) > 1$, $n > 1$, then the control system is controllable over a path-connected, open dense subset of the configuration space (comprised of configurations with fixed centroid). We assume here without loss of generality that G is weakly connected and that $N > n$. In case G is not weakly connected, one can analyze the weakly connected components independently using the results of this paper, and in case $N \leq n$, one can see that the dynamics (1) evolves in a proper affine subspace of \mathbb{R}^n with its dimension less than N . Thus,

one can use the results of this paper, after a simple change of variables, to study that case as well.

One of main contributions of this paper is to identify a class of weakly connected directed graphs for which the system (1) is controllable. In particular, we will establish a relation between the geometry of formations, the structure of the underlying network topology, and the controllability of the formation control system. This paper expands on the preliminary version [13] by, among others, providing an analysis of the formation control system (1) with time-varying graphs, a finer description of their reachable sets and proofs that were omitted.

Following this introduction, the remainder of the paper is organized as follows. In the next section, we introduce some definitions and state the main theorem. We also derive properties of the configuration space of the formation control system; in particular, we identify an open dense subset of the configuration space where system (1) is controllable. We obtain a necessary and sufficient condition for this open dense subset to be path-connected. Next, we introduce the matrix Lie algebra \mathcal{A} of zero row-sum matrices and show how to relate the graph closure of G to the Lie algebraic closure of a naturally defined subspace of \mathcal{A} . In section 3, we compute the Lie brackets of control vector fields and prove the controllability of system (1) by verifying the Lie algebra rank condition. We summarize and provide future directions in the last section.

II. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

A. Digraphs and their strong component decompositions

Let $G = (V, E)$ be a directed graph (or simply *digraph*) of N vertices with $V = \{v_1, \dots, v_N\}$ the set of vertices and E the set of edges. We denote by $v_i \rightarrow v_j$ a directed edge in G from v_i to v_j . We call a digraph G **weakly connected** if the undirected graph obtained by ignoring the orientation of the edges is connected [14]. The digraph G is **strongly connected** if for any pair of vertices v_i and v_j , there is a path in G from v_i to v_j . We say that $G_i = (V_i, E_i)$ is a subgraph of G if $V_i \subset V$ and $E_i \subset E$. We call two subgraphs G_i and G_j **disjoint** if $V_i \cap V_j = \emptyset$. Furthermore, we say that G_i is **induced by** V_i if E_i contains all edges in E that connect vertices in V_i , i.e.

$$E_i := \{v_k \rightarrow v_l \in E \mid v_k, v_l \in V_i\}.$$

Definition 1 (Strong component decomposition). *We say that the subgraphs $G_i = (V_i, E_i)$, $1 \leq i \leq q$, form a **strong component decomposition** of G if*

- 1) *Each subgraph G_i , for $1 \leq i \leq q$, is induced by V_i , and the G_i 's are pairwise disjoint.*
- 2) *The V_i 's partition the vertex set V : $\sqcup_{i=1}^q V_i = V$.*

We are interested in strong component decompositions with the least possible number of subgraphs. We call such a decomposition **coarse**. The following lemma shows that there is a *unique* coarse strong component decomposition of a weakly connected digraph.

Lemma 1. *Let $G = (V, E)$ be a weakly connected digraph. There is a unique strong component decomposition (SCD) of smallest cardinality.*

Proof. We prove Lemma 1 by contradiction. Let q be the minimal number of subsets in a SCD. Suppose that there are two distinct sets of subgraphs, $\{G_1, \dots, G_q\}$ and $\{G'_1, \dots, G'_q\}$, and that they both are SCDs of G with q subsets. Then, after a relabeling of the subgraphs, we can assume that $V_1 \not\subseteq V'_1$ and $V'_1 \not\subseteq V_1$. Indeed, if this does not hold, then the two SCDs are identical. We now collect sets V_i 's that intersect V'_1 ; define S as follows: if $i \in S$, then $V_i \cap V'_1 \neq \emptyset$. Since the V_i 's are disjoint and $V'_1 \not\subseteq V_1$, we need at least two sets to cover V'_1 . Hence, the cardinality of S is at least two.

Now set $V^* := \sqcup_{i \in S} V_i$, and let $G^* = (V^*, E^*)$ be the digraph induced by V^* . We show below that G^* is strongly connected. Note that if it is the case, then we can obtain a SCD of G whose cardinality is strictly smaller than q : indeed, the subgraphs G_j 's, for $j \notin S$, together with G^* form a SCD of G . Moreover, the number of the subsets of the SCD is strictly smaller than q since the cardinality of S is at least 2. We thus derive a contradiction, and hence, conclude that there is a unique SCD with smallest number of subsets.

We now show that G^* is strongly connected. First, we show that for any $v_k \in V'_1$ and any $v_{l_i} \in V_i$, for $i \in S$, there is a path γ_{kl_i} from v_k to v_{l_i} and a path $\gamma_{l_i k}$ from v_{l_i} to v_k . Pick a vertex $v_{k_i} \in V_i \cap V'_1$. Such a vertex exists by definition of S . Then, there is a path from v_k to v_{k_i} in G'_1 and a path from v_{k_i} to v_{l_i} in G_i . Using these two paths, we can obtain a path γ_{kl_i} from v_k to v_{l_i} . Using the same argument, we can also obtain a path $\gamma_{l_i k}$ from v_{l_i} to v_k . But then, since G'_1 and the G_i 's are strongly connected, we can use the paths $\gamma_{l_i k}$ and γ_{kl_i} to obtain a path from v_{l_i} to v_{l_j} . Using again the fact that the G_i 's are strongly connected, we obtain a path from any vertex in G_i to any vertex in G_j and vice-versa. Thus, we have shown that G^* is strongly connected. \square

The coarse strong component decomposition of a weakly connected digraph G induces an acyclic digraph, with the vertices representing the components of G and edges representing the flows between the components. Precisely, we have the following definition:

Definition 2 (Skeleton digraph). *Let G be a weakly connected digraph, and let G_1, \dots, G_q form the coarse strong component decomposition of G . Define an acyclic digraph $H = (W, F)$ with q vertices as follows: there is an edge $w_i \rightarrow w_j$ in H if and only if there is an edge $v_i \rightarrow v_j$ in G with v_i a vertex in G_i and v_j a vertex in G_j . The acyclic digraph H will be referred as the **skeleton digraph** of G .*

The digraph H defines a partial order on its vertices: we say w_j is greater than w_i , or simply $w_j \succ w_i$, if there is a path from w_i to w_j in H . We say a vertex w_i of H is **maximal** for

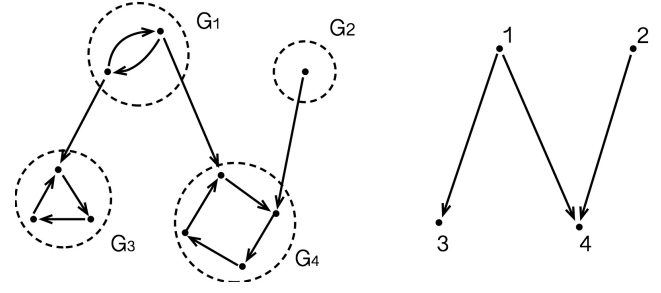


Fig. 1. By applying the coarse strong component decomposition to the weakly connected graph G on the left, we get four strongly connected subgraphs as G_1, \dots, G_4 . The skeleton digraph H of G is given on the right hand side of the figure. The maximal set W_+ of the skeleton of G is given by $\{3, 4\}$

the partial order \succ if there does not exist a vertex w_j such that $w_j \succ w_i$. Denote by $W_+ \subseteq W$ the set of maximal elements and refer to it as the **maximal set** of the skeleton of G . By definition, if $w_i \in W_+$, then it does not have any outgoing neighbors of w_i . Also, we note that for any $w_i \in W - W_+$, there is at least a $w_j \in W_+$, together with a path from w_i to w_j .

B. Configuration space

Given a formation of N agents in \mathbb{R}^n , with states x_1, \dots, x_N respectively, we set $p = (x_1, \dots, x_N) \in \mathbb{R}^{nN}$. We call p the **configuration** of the system and $P := \mathbb{R}^{nN}$ the **configuration space** of the system. Let Q be a subset of P . We say that Q is **path-connected** if for any two configurations $p_0, p_1 \in Q$, there is a continuous function $p(t) : [0, 1] \rightarrow P$ with $p(0) = p_0$ and $p(1) = p_1$ such that the image of $p(t)$ lies in Q . We say that Q is **disconnected** if it is not path-connected. We now define what it means for a system to be approximately path-controllable:

Definition 3 (Approximate path-controllability). *Let Q be a path-connected set. We say that system (1) is **approximately path-controllable** over Q if for any $T > 0$, any arbitrary continuous curve $\hat{p} : [0, T] \rightarrow Q$ and any tolerance $\epsilon > 0$, there are integrable functions $u_{ij}(p, t)$'s such that the solution $p(t)$ of system (1), from any initial condition $p(0)$ with $\|p(0) - \hat{p}(0)\| < \epsilon$, satisfies*

$$\|\hat{p}(t) - p(t)\| < \epsilon$$

for all $t \in [0, T]$.

We denote by \mathbf{u} the ensemble of controls u_{ij} 's, and let $\mathbf{u}[0, T]$ be the function \mathbf{u} over the time interval $[0, T]$. We now state the main theorem of this paper.

Theorem 1. *Let G be a weakly connected digraph, and let $G = \{G_1, \dots, G_q\}$ be the strong component decomposition of G . If for each $w_i \in W_+$ we have*

$$|G_i| > (n + 1),$$

then system (1) is approximately path-controllable over a path-connected, open dense subset of P .

The path-controllability of system (1) is established below by verifying the Lie algebra rank condition of the control vector fields. Precisely, we show that the Lie algebra rank condition is satisfied over a path-connected, open dense subset of P as long as the graph G satisfies the assumption of Theorem 1. The same proof technique can actually be used to handle time-varying graphs. Let $G(t) = (V, E(t))$ be a right-continuous, time-varying graph. We call a **switching time** a time t_i such that $\lim_{t \rightarrow t_i, t < t_i} G(t) \neq G(t_i)$. We obtain as a corollary of Theorem 1 the following result:

Corollary 2. *Let $G(t)$ be a right-continuous time-varying graph such that for any finite time interval, $G(t)$ has a finite number of switching times. Suppose that for each $t \geq 0$, the graph $G(t)$ satisfies the assumption of Theorem 1. Then, the formation control system (1) is approximately path-controllable over a path-connected, open dense subset of P .*

Proof. Let $\hat{p} : [0, T] \rightarrow Q$ be the path we want the system to follow and let t_1, \dots, t_m be the switching times of $G(t)$. We construct an admissible $\mathbf{u}[0, T]$ as follows. Given a graph $G(0)$ and an initial configuration $p(0)$ that satisfies $\|p(0) - \hat{p}(0)\| < \epsilon$, we know from Theorem 1 that there exists $\mathbf{u}_1[0, T]$ such that system (1) approximates \hat{p} over $[0, T]$. We use this control until the first switching time: $\mathbf{u}[0, t_1] = \mathbf{u}_1[0, t_1]$. It follows that $\|p(t_1) - \hat{p}(t_1)\| < \epsilon$. We can thus apply Theorem 1 but with graph $G(t_1)$ to obtain a control law $\mathbf{u}_2[t_1, T]$ that steers the system from $p(t_1)$ along a trajectory $p(t)$ such that $\|p(t) - \hat{p}(t)\| < \epsilon$. As before, we let $\mathbf{u}[t_1, t_2] = \mathbf{u}_2[t_1, t_2]$. Note that implementing the control \mathbf{u} over the time interval $[0, t_2]$ yields a trajectory $p(t)$ within ϵ tolerance of \hat{p} over that interval. Repeating this procedure a finite number of times yields a control \mathbf{u} that approximates \hat{p} as required. \square

C. Non-degenerate configurations

Let p be a configuration of m agents x_1, \dots, x_m in \mathbb{R}^n . We say that r_p is the **rank** of p , if there is no affine subspace of dimension $(r_p - 1)$ that contains p . Equivalently, it is the dimension of the linear span of $\{x_i - x_1, \dots, x_i - x_m\}$ for some (and hence, any) $i = 1, \dots, m$.

Definition 4 (Non-degenerate configuration). *We say that the configuration $p = (x_1, \dots, x_m) \in \mathbb{R}^{nm}$, $x_i \in \mathbb{R}^n$, is **non-degenerate** in \mathbb{R}^n if there is no proper affine subspace of \mathbb{R}^n containing x_1, \dots, x_m . Equivalently, p is non-degenerate if p is of full rank, i.e., the linear span of the vectors $\{x_i - x_1, \dots, x_i - x_m\}$ is \mathbb{R}^n .*

We note that if p is non-degenerate, then the number of agents has to be strictly greater than n . If $G = (V, E)$ is a digraph with N vertices, a configuration $p \in P$ can be viewed as an embedding of the graph G in \mathbb{R}^n by assigning vertex v_i to x_i . We call the pair (G, p) a **framework**. Let (G, p) be a framework with G weakly connected, and let G_1, \dots, G_k form the coarse strong component decomposition of G . We denote by (G_i, p_i) , with $p_i \in \mathbb{R}^{n|G_i|}$, the framework obtained from (G, p) by only considering vertices and edges of G_i . We refer to p_i the **sub-configuration** associated with G_i , and similarly

denote by r_{p_i} the rank of p_i . Let Q be the subset of P defined as follows:

$$Q := \{p \in P \mid r_{p_i} = n, \forall w_i \in W_+\} \quad (2)$$

where we recall that W_+ is the maximal set of the skeleton of G . It should be clear that Q is an open subset of P . We now show that Q is also dense in P , and even path-connected under some mild assumptions.

Proposition 3. *Under the assumption of Theorem 1, the set Q , defined in (2), is an open dense, path-connected subset of P .*

We can decompose the set P into disjoint components as follows. Fix an integer $k \geq 0$ and define the set

$$P^k := \{p \in P \mid r_p = k\}.$$

It is easy to see that each P^k is nonempty, for $k = 0, \dots, n$. Note that the set P^0, \dots, P^n are pairwise disjoint, and they form a decomposition of P :

$$P = \sqcup_{k=0}^n P^k.$$

Furthermore, for each $k = 1, \dots, n$, we have

$$\overline{P^k} - P^k = \sqcup_{l=0}^{k-1} P^l \quad (3)$$

where $\overline{P^k}$ is the closure of P^k in \mathbb{R}^{nN} . For $k = 0$, we have

$$\overline{P^0} = P^0 = \{p = (x_1, \dots, x_n) \in P \mid x_1 = \dots = x_n\}.$$

Note that P^n is the set of non-degenerate configurations. Since P^n is an open set in P , it is a smooth submanifold of P . Also, from (3), the set P^n is dense in P , i.e.,

$$\overline{P^n} = P. \quad (4)$$

The union of the other sets P^0, \dots, P^{n-1} is the set of degenerate configurations. Define

$$d_k := -k^2 + k(N + n - 1) + n. \quad (5)$$

We now have the following fact:

Lemma 4. *Each P^k is a smooth submanifold of P , and $\dim P^k = d_k$.*

We refer to the Appendix for a complete proof of Lemma 4. For each $k = 0, \dots, n$, the codimension of P^k in P is by convention defined to be:

$$\text{codim } P^k := \dim P - \dim P^k.$$

We now establish the following inequalities:

Lemma 5. *Suppose that $N > n$. Then,*

$$\text{codim } P^k \geq N - n$$

for all $k = 0, \dots, n - 1$.

Proof. We prove the result by evaluating the value of d_k . On one-hand, from (5), we have that d_k is a quadratic function in the variable k , and achieves its maximum at $(N + n - 1)/2$. Since $N > n$, we have

$$\frac{1}{2}(N + n - 1) \geq n,$$

and thus d_k is a strictly monotonically increasing function in k for $k = 0, \dots, n-1$, from which we conclude that

$$d_0 < \dots < d_{n-1}.$$

On the other hand, we have that

$$d_{n-1} = nN - N + n.$$

Thus, the codimension of P^k satisfies

$$\text{codim } P^k \geq \text{codim } P^{n-1} = N - n.$$

for all $k = 0, \dots, n-1$. \square

Let $G_i = (V_i, E_i)$, for $i = 1, \dots, q$, form the coarse strong component decomposition of G . Let $V_{-i} := V - V_i$, and G_{-i} be the subgraph of G induced by V_{-i} . Let P_i and P_{-i} be the sets of sub-configurations associated with G_i and G_{-i} , respectively. Let N_i be the number of vertices of G_i . We have that $\dim P_i = n \times N_i$ and $\dim P_{-i} = n \times (N - N_i)$. Similarly, we define P_i^k to be the set of rank- k sub-configurations associated with G_i . We are now in the position to prove Proposition 3.

Proof of Proposition 3. Recall that W_+ , defined in section II-A, is the maximal set of the skeleton of G . We first show that Q is an open dense subset of P , and then show that Q is path-connected. Following the definition of Q (in (2)), we can write

$$Q = \bigcap_{w_i \in W_+} (P_i^n \times P_{-i}) \quad (6)$$

From the assumption of Theorem 1, we have $N_i - n > 1$ for all $w_i \in W_+$. Then, each P_i^n is a nonempty open set in P_i , and from (4), we have that $\tilde{P}_i^n = P_i$ for all $w_i \in W_+$. So then, each $P_i^n \times P_{-i}$ is an open dense subset of P , and so is Q . This holds because a finite intersection of open dense subsets is still open and dense.

We now show that Q is path-connected. For each $w_i \in W_+$, define \tilde{P}_i to be the set of degenerate configurations in P_i . Then, \tilde{P}_i can be expressed as follows:

$$\tilde{P}_i = \bigcup_{k=0}^{n-1} P_i^k.$$

Since $P_i = \tilde{P}_i \sqcup P_i^n$, from (6), we can express Q as follows:

$$Q = P - \bigcup_{w_i \in W_+} (\tilde{P}_i \times P_{-i}).$$

We now show that each $\tilde{P}_i \times P_{-i}$, for $w_i \in W_+$, is a finite union of smooth submanifolds of P and that the codimension of each submanifold is strictly greater than 1. First, note that

$$\tilde{P}_i \times P_{-i} = \bigcup_{k=0}^{n-1} (P_i^k \times P_{-i}).$$

By Lemma 4, each P_i^k is a submanifold of P_i . Furthermore, by Lemma 5, the codimension of P_i^k in P_i satisfies

$$\text{codim } P_i^k := \dim P_i - \dim P_i^k \geq N_i - n$$

for all $k = 0, \dots, n-1$. Using the fact that $P = P_i \times P_{-i}$, we have

$$\text{codim}(P_i^k \times P_{-i}) = \text{codim } P_i^k,$$

and hence,

$$\text{codim}(P_i^k \times P_{-i}) \geq (N_i - n) > 1.$$

We have thus proved that each $\tilde{P}_i \times P_{-i}$, for $w_i \in W_+$, is a finite union of smooth submanifolds of P , and the codimension of each submanifold is strictly greater than 1. Since removing from a Euclidean space a finite union of smooth submanifolds of codimensions at least two does not render it disconnected, the result is proved. \square

Proposition 3 shows that the assumption of Theorem 1 is sufficient for Q to be a nonempty path-connected, open dense subset. We show below that it is also necessary for Q . Recall that P_i^k is the set of rank- k sub-configurations associated with G_i , and in particular, P_i^n is the set of non-degenerate sub-configurations associated with G_i . We first establish the following fact.

Proposition 6. *Let W_+ be the maximal set of the skeleton of G . Suppose that there exists a $w_i \in W_+$ such that $N_i - n \leq 1$. Then, the following two results hold:*

1. *If $N_i - n \leq 0$, then the set Q is empty.*
2. *If $N_i - n = 1$, then P_i^n has two connected components.*

Proof. First, observe that if a configuration is non-degenerate in \mathbb{R}^n , it contains at least $(n+1)$ agents. Thus, if $N_i \leq n$, then there does not exist a non-degenerate sub-configuration associated with G_i , and hence, Q is empty.

We now assume that $N_i - n = 1$, and prove that P_i^n has two connected components. Without loss of generality, we assume that p_i is formed by agents x_1, \dots, x_{n+1} . For each $p_i \in P_i^n$, define a matrix as follows:

$$A_{p_i} := (x_2 - x_1, \dots, x_{n+1} - x_1) \in \mathbb{R}^{n \times n}.$$

Since p_i is non-degenerate, A_{p_i} is invertible. Let $\text{GL}(n)$ be the set of n -by- n invertible matrices, and denote by

$$f : P_i^n \longrightarrow \text{GL}(n)$$

the smooth map sending p_i to A_{p_i} . Note that f is surjective and open; indeed, for a matrix $A \in \text{GL}(n)$, with column vectors $a_1, \dots, a_n \in \mathbb{R}^n$, we have

$$f^{-1}(A) = \{(x_1, x_1 + a_1, \dots, x_1 + a_n) \mid x_1 \in \mathbb{R}^n\}.$$

Thus, we have

$$P_i^n = f^{-1} \text{GL}(n). \quad (7)$$

It is well known that $\text{GL}(n)$ has two connected components: the matrices with positive determinant and the ones with negative determinant. Thus, following (7), we conclude that P_i^n has two connected components: the p_i 's with $\det(A_{p_i}) > 0$, and the ones with $\det(A_{p_i}) < 0$. This completes the proof. \square

Following Proposition 6, we have the following corollary:

Corollary 7. *Let W_+ be the maximal set of the skeleton of G , and Q be defined in (2). Suppose that there exists a $w_i \in W_+$ such that $N_i - n \leq 1$. Then, Q is disconnected.*

Proof. First, from Proposition 6, we know that P_i^n has two connected components, and hence, so does $P_i^n \times P_{-i}$. On the other hand, from (6), we have that

$$Q = (P_i^n \times P_{-i}) - \bigcup_{j \in W_+ - \{i\}} (\tilde{P}_j \times P_{-j})$$

from which it follows that the set Q is disconnected. \square

We conclude this section by applying Propositions 3 and 6 to a special case where the digraph G is strongly connected. In this case, the skeleton digraph H of G is comprised of only one vertex, and hence, Q is the set of non-degenerate configurations in \mathbb{R}^n . We have the following fact:

Corollary 8. *Let G be a strongly connected graph with N vertices. Then, the following three properties hold:*

1. *If $N < n + 1$, then Q is empty.*
2. *If $N = n + 1$, then Q is open dense in P , and it has two connected components.*
3. *If $N > n + 1$, then Q is an open dense, path-connected subset of P .*

The first two properties in the corollary follow from Proposition 6, and the last one follows from Proposition 3.

D. Non-degenerate sub-configurations

If p is a non-degenerate configuration in \mathbb{R}^n , then there exists at least one set of $(n + 1)$ agents such that the sub-configuration formed by these $(n + 1)$ agents is non-degenerate in \mathbb{R}^n . We establish below a tighter lower bound on the number of non-degenerate sub-configurations of p .

To this end, let $p = (x_1, \dots, x_{n+1})$ be a configuration in \mathbb{R}^n associated with a digraph $G = (V, E)$ with $(n + 1)$ vertices. For each $i \in V$, denote by S_i the affine subspace of \mathbb{R}^n of lowest dimension that contains the n vectors x_j 's, for $j \in V - \{i\}$. Note that if p is non-degenerate, then each S_i , for $i \in V$, is a hyperplane in \mathbb{R}^n , i.e., $\dim S_i = n - 1$. Similarly, for any proper subset $V' \subset V$, we define $S_{V'}$ as the affine subspace of \mathbb{R}^n of lowest dimension that contains vectors x_i 's, for all $i \in V - V'$. We now establish the following result which relates $S_{V'}$ to the intersection of the hyperplanes S_i 's:

Proposition 9. *Let $p = (x_1, \dots, x_{n+1})$ be a non-degenerate configuration in \mathbb{R}^n associated with a digraph $G = (V, E)$ with $(n + 1)$ vertices. Let V' be a proper subset of V . Then,*

$$S_{V'} = \bigcap_{i \in V'} S_i.$$

Proof. Without loss of generality, we assume that $V' = \{1, \dots, k\}$ with $k \leq n$. For each $i \in V'$, denote by $\text{span}\{x_j - x_{n+1} \mid j \neq i\}$ the linear subspace of \mathbb{R}^n spanned by the $(n - 1)$ vectors $\{x_j - x_{n+1} \mid j \neq i\}$. So then,

$$S_i = x_{n+1} + \text{span}\{x_j - x_{n+1} \mid j \neq i\}.$$

Since p is non-degenerate, the n vectors $\{x_i - x_{n+1} \mid 1 \leq i \leq n\}$ are linearly independent. Thus,

$$\bigcap_{i \in V'} \text{span}\{x_j - x_{n+1} \mid j \neq i\} = \text{span}\{x_j - x_{n+1} \mid j \notin V'\},$$

and hence, it follows that

$$\bigcap_{i \in V'} S_i = x_{n+1} + \text{span}\{x_j - x_{n+1} \mid j \notin V'\} = S_{V'}.$$

\square

We obtain a corollary of Proposition 9 as follows:

Corollary 10. *Let $p = (x_1, \dots, x_{n+1})$ be a non-degenerate configuration in \mathbb{R}^n associated with a digraph $G = (V, E)$ with $(n + 1)$ vertices. Then, for each $i \in V$, we have*

$$\bigcap_{j \neq i} S_i = \{x_i\}.$$

Proof. Let $V' := V - \{i\}$, then $S_{V'} = \{x_i\}$; indeed, the affine subspace in \mathbb{R}^n of lowest dimension that contains x_i is the singleton $\{x_i\}$. Then, following Proposition 9, we have $\bigcap_{i \in V'} S_i = S_{V'} = \{x_i\}$. \square

We now establish the following fact.

Proposition 11. *Let $p = (x_1, \dots, x_{n+1})$ be a non-degenerate configuration in \mathbb{R}^n . Then, for any vector $x \in \mathbb{R}^n$, there exist n vectors $\{x_{i_1}, \dots, x_{i_n}\} \subset \{x_1, \dots, x_{n+1}\}$ such that these n vectors together with x form a non-degenerate configuration in \mathbb{R}^n .*

Proof. We prove the result by contradiction. Assume that there is a vector x in \mathbb{R}^n such that there does not exist a set of n vectors $\{x_{i_1}, \dots, x_{i_n}\}$ out of $\{x_1, \dots, x_{n+1}\}$ such that $x, x_{i_1}, \dots, x_{i_n}$ form a non-degenerate configuration in \mathbb{R}^n .

Recall that S_i is the affine subspace of \mathbb{R}^n of lowest dimension that contains the n vectors x_j 's, for $j \neq i$. Since p is a non-degenerate configuration, by Corollary 10, we have

$$\bigcap_{j \neq i} S_j = \{x_i\}.$$

So then, we have

$$\bigcap_{i=1}^{n+1} S_i = \bigcap_{i=1}^{n+1} \bigcap_{j \neq i} S_j = \bigcap_{i=1}^{n+1} \{x_i\} = \emptyset.$$

On the other hand, each S_i has to contain the vector x because otherwise the $(n + 1)$ vectors $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}, x$ form a non-degenerate configuration in \mathbb{R}^n . Thus,

$$x \in \bigcap_{i=1}^{n+1} S_i = \emptyset$$

which is a contradiction. This completes the proof. \square

We obtain as a corollary a lower bound on the number of non-degenerate sub-configurations of p :

Corollary 12. *Let $p \in \mathbb{R}^{nN}$ be a non-degenerate configuration with $N > n$. Then, there are at least $(N - n)$ sub-configurations of $(n + 1)$ agents that are non-degenerate in \mathbb{R}^n .*

E. Lie algebra of zero row-sum matrices

Definition 5 (Zero row-sum matrices). Denote by $\mathbf{1}$ the vector of \mathbb{R}^N with all entries one. We say that a matrix $A \in \mathbb{R}^{N \times N}$ is a **zero row-sum matrix** if $A\mathbf{1} = 0$. We denote by \mathbb{A} the vector space of such matrices.

It is easy to verify that the commutator or Lie bracket of two zero row-sum matrices is also a zero row-sum matrix. Hence, the vector space \mathbb{A} is a Lie algebra. We derive here some properties of the Lie algebra of zero row-sum matrices that are needed in the proof of the main Theorem.

Let e_1, \dots, e_N be the canonical basis of \mathbb{R}^N . Let $A_{ij} \in \mathbb{A}$ be defined as follows:

$$A_{ij} := -e_i e_i^\top + e_i e_j^\top.$$

Note that the matrix A_{ij} is the negative of the Laplacian matrix of a digraph with N vertices and only one edge, namely $v_i \rightarrow v_j$. For a digraph $G = (V, E)$, define a set of zero-row sum matrices as follows:

$$A_G := \{A_{ij} \in \mathbb{R}^{N \times N} \mid v_i \rightarrow v_j \in E\}$$

It is easy to see that matrices in A_G are linearly independent. We denote by \mathbb{A}_G the vector space spanned by elements in A_G . Further, we introduce the following definitions:

Definition 6 (Lie algebra closure of a vector space of matrices). Given a vector space $\mathbb{A} \subset \mathbb{R}^{N \times N}$ of matrices, we denote by $\overline{\mathbb{A}}$ the **Lie algebra closure** of \mathbb{A} , defined as the vector space of least dimension in $\mathbb{R}^{N \times N}$ which contains \mathbb{A} and is closed under the matrix Lie bracket.

Definition 7 (Transitive closure of a digraph). Given a digraph G , we denote by \overline{G} the **transitive closure** of G : \overline{G} has the same vertex set as G and there is an edge $v_i \rightarrow v_j$ in \overline{G} if and only if there is a path from v_i to v_j in G .

For illustration of the transitive closure of a digraph, we refer to the example depicted in Figure 2.

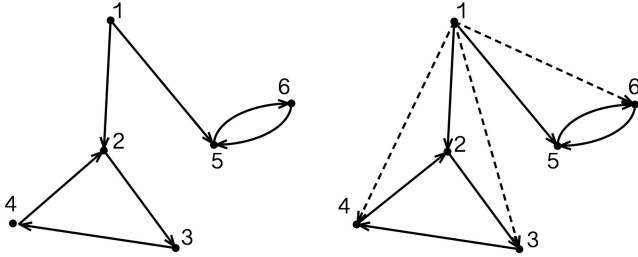


Fig. 2. The digraph on the right hand side of this figure is the transitive closure of the digraph on the left hand side.

Our goal in this section is to evaluate $\overline{\mathbb{A}_G}$ for G , a weakly connected graph. In particular, we establish the following result.

Proposition 13. Let G be a weakly connected digraph, and \overline{G} be its transitive closure. Let $\overline{\mathbb{A}_G}$ be the Lie algebra closure of \mathbb{A}_G . Then,

$$\overline{\mathbb{A}_G} = \mathbb{A}_{\overline{G}}.$$

Proposition 13 relates the Lie algebra closure of a set of Laplacian matrices to the transitive closure of a digraph G . To prove Proposition 13, we first show that strong component decompositions and transitive closures commute:

Lemma 14. Let G be a weakly connected digraph, and H be the associated skeleton digraph. Then, the following holds:

1. If G_1, \dots, G_q form the coarse strong component decomposition of G , then $\overline{G}_1, \dots, \overline{G}_q$ form the coarse strong component decomposition of \overline{G} . In particular, each \overline{G}_i , for $i = 1, \dots, q$, is a complete graph.
2. Let \overline{H} be the transitive closure of H . Then, \overline{H} is the skeleton digraph of \overline{G} .

We prove the Lemma in the Appendix.

We now evaluate the Lie brackets of matrices in A_G .

Lemma 15. Let (i, j) and (i', j') be two pairs of positive integers with $1 \leq i \neq j \leq n$ and $1 \leq i' \neq j' \leq n$. Then, the following three properties hold:

1. If $i \neq i'$ and $j \neq j'$, then

$$[A_{ij}, A_{i'j'}] = 0.$$

2. If $i = i'$, then

$$[A_{ij}, A_{ij'}] = A_{ij} - A_{ij'}.$$

3. If $j = j'$, then

$$[A_{ij}, A_{i'j}] = A_{i'j} - A_{ij}.$$

We omit the proof of the above, as the result follows directly from computations. We mention that similar results have been obtained in [12] and [15]. As a corollary of Lemma 15, we have the following fact:

Corollary 16. If $v_i \rightarrow v_j$ is an edge of \overline{G} , then the matrix A_{ij} is contained in $\overline{\mathbb{A}_G}$.

Proof. Let $v_i \rightarrow v_j$ be an edge of \overline{G} ; then by the definition of transitive closure, there is a path from v_i to v_j in G . Suppose that the path is of length k , and we express it as follows:

$$v_{i_0} \rightarrow \dots \rightarrow v_{i_k}$$

with $i_0 = i$ and $i_k = j$. We now show that $A_{i_0 i_k} \in \overline{\mathbb{A}_G}$. The proof is carried out by induction on the length k . For the base case, we have $k = 2$; then $v_i \rightarrow v_j$ is an edge of G . Thus, A_{ij} is in \mathbb{A}_G . For the inductive step, suppose that the Lemma holds for $(k-1)$, and we prove for k . First, by the induction hypothesis, the matrix $A_{i_0 i_{k-1}}$ is in $\overline{\mathbb{A}_G}$. Then, from Lemma 15, we have

$$A_{i_0 i_k} = [A_{i_0 i_{k-1}}, A_{i_{k-1} i_k}] + A_{i_0 i_{k-1}} \in \overline{\mathbb{A}_G}.$$

This completes the proof. We also illustrate the generating procedure in Figure 3. \square

From Corollary 16, it should be clear now that $\overline{\mathbb{A}_G}$ contains the vectors space $\mathbb{A}_{\overline{G}}$. Hence, Proposition 13 will be established after we prove the following Lemma.

Lemma 17. The vector space $\mathbb{A}_{\overline{G}}$ is closed under the Lie bracket.

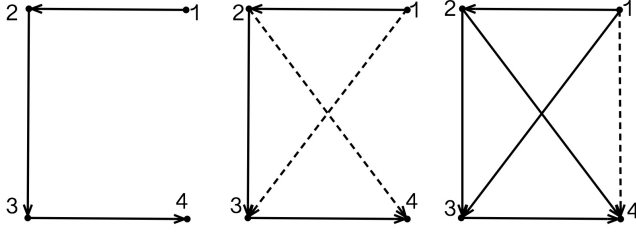


Fig. 3. We illustrate here the generating process of Lie brackets of A_{ij} 's. In this example, we start from A_{12}, A_{23}, A_{34} . Taking Lie brackets, we get $A_{13} = [A_{12}, A_{23}] + A_{12}$ and $A_{24} = [A_{23}, A_{34}] + A_{23}$. Taking the Lie bracket of the original matrix and the newly obtained ones, we get $A_{14} = [A_{13}, A_{34}] + A_{13} = [A_{12}, A_{24}] + A_{12}$.

Proof. We know that the set $A_{\bar{G}}$ is a basis of the vector space $\mathbb{A}_{\bar{G}}$. So it suffices to show that for any two matrices A_{ij} and $A_{i'j'}$ in $A_{\bar{G}}$, the Lie bracket $[A_{ij}, A_{i'j'}]$ is a linear combination of matrices in $A_{\bar{G}}$. This is a direct consequence of Lemma 15; indeed, in case $i' \neq i$ and $j' \neq j$, we have

$$[A_{ij}, A_{i'j'}] = 0,$$

in case $i' = i$, then

$$[A_{ij}, A_{ij'}] = A_{ij} - A_{ij'},$$

and in case $j' = j$, then

$$[A_{ij}, A_{ij'}] = A_{ij'} - A_{ij}$$

with $A_{ij'}$ in $A_{\bar{G}}$ because $v_i \rightarrow v_j \rightarrow v_{j'}$ is a path in G which implies that $v_i \rightarrow v_{j'}$ is an edge of \bar{G} . \square

Lemma 17 implies that the vector space $\mathbb{A}_{\bar{G}}$ contains the Lie algebra closure of \mathbb{A}_G . Proposition 13 then follows by combining Corollary 16 and Lemma 17.

III. LIE ALGEBRA OF CONTROL VECTOR FIELDS

We now prove the controllability of system (1) by verifying the Lie algebra rank condition over the path-connected, open dense set Q defined in (2). We first rewrite system equation (1) into a matrix form which makes it simpler to evaluate the Lie brackets of the control vector fields. To this end, we re-order the entries of the vector p as follows. Let x_i^j be the j -th coordinate of agent i , and let

$$x^j := (x_1^j, \dots, x_N^j)$$

be a vector in \mathbb{R}^N collecting the j -th coordinate of all agents. In the remainder of this section, the configuration vector p is taken as

$$p = (x^1, \dots, x^n).$$

Let A be an N -by- N matrix, and let

$$D(A) := \text{Diag}(A, \dots, A)$$

be a block-diagonal matrix with A repeated n times. Then, with the notations above, system (1) can be expressed as

$$\dot{p} = \sum_{i \rightarrow j \in E} u_{ij} D(A_{ij}) p$$

with u_{ij} 's the scalar controls. Note that the control system above is in a standard affine control form [16], [17] with

$$g_{ij}(p) := D(A_{ij})p, \quad i \rightarrow j \in E$$

the control vector fields.

Definition 8 (Lie algebra rank condition). *Let \mathcal{L} be the Lie algebra generated by the control vector fields g_{ij} 's. Let \mathcal{L}_p be the vector space obtained by evaluating the elements of \mathcal{L} at p . We say that \mathcal{L}_p satisfies the **Lie algebra rank condition** if*

$$\dim \mathcal{L}_p = \dim P = n \times N.$$

We now establish the following result.

Proposition 18. *Let Q be the path-connected, open dense subset of P defined by (2). Then, under the assumption of Theorem 1, \mathcal{L}_p satisfies the Lie algebra rank condition for all $p \in Q$.*

We prove Proposition 18 below. First, note that for any two matrices A_{ij} and $A_{i'j'}$ in $\mathbb{A}_{\bar{G}}$, we have

$$[D(A_{ij}), D(A_{i'j'})] = D([A_{ij}, A_{i'j'}]).$$

Thus, by Proposition 13, we have

$$\mathcal{L}_p = \{D(A)p \mid A \in \mathbb{A}_{\bar{G}}\} \quad (8)$$

It suffices to show that there are $(n \times N)$ linearly independent vectors in \mathcal{L}_p .

We start the proof with a special case. Consider a strongly connected digraph G with N vertices, with $N = n + 1$. From Corollary 8, the set Q is an open dense subset of P . We now establish the following result.

Lemma 19. *Let G be a strongly connected digraph with $(n + 1)$ vertices. Then, for any $p \in Q$, we have*

$$\dim \mathcal{L}_p = n(n + 1).$$

Proof. Since G is strongly connected, the transitive closure \bar{G} is a complete graph. Hence there are $n(n + 1)$ matrices in $A_{\bar{G}}$. We thus need to show that the vectors $\{D(A_{ij})p \mid i \rightarrow j \in \bar{G}\}$ are linearly independent. This is equivalent to showing that if $D(A)p = 0$ for some $A \in A_{\bar{G}}$, then $A = 0$. To do so, we first introduce a matrix as follows:

$$X_e := (\mathbf{1}, x^1, \dots, x^n) \in \mathbb{R}^{(n+1) \times (n+1)}$$

where $\mathbf{1}$ is the vector of all ones in \mathbb{R}^{n+1} . We show that the matrix X_e is nonsingular if $p \in Q$; indeed, consider the following elementary row operation on X_e : let

$$\tilde{X}_e := R X_e$$

with R given by

$$R := \begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 \end{pmatrix}.$$

Then, by computation, we have

$$\tilde{X}_e = \begin{pmatrix} 1 & x_1^\top \\ 0 & (x_2 - x_1)^\top \\ \vdots & \vdots \\ 0 & (x_{n+1} - x_1)^\top \end{pmatrix}.$$

Since $p \in Q$, p is non-degenerate. Hence, the n vectors $\{x_2 - x_1, \dots, x_{n+1} - x_1\}$ are linearly independent. This shows that \tilde{X}_e is nonsingular, and so is X_e . On the other hand, if $D(A)p = 0$, then $Ax^i = 0$ for all $i = 1, \dots, n$. Furthermore, $A\mathbf{1} = 0$ since A is a zero-row sum matrix. Thus, we have $AX_e = 0$. Since X_e is nonsingular, we have $A = 0$. This completes the proof. \square

We now prove Proposition 18.

Proof of Proposition 18. We prove the result by directly constructing a set of $(n \times N)$ linearly independent vectors in \mathcal{L}_p . Suppose that G_1, \dots, G_q form the coarse strong component decomposition of G . Let N_i be the number of vertices of G_i . Without loss of generality, we label the vertices of G so that the first N_1 vertices are of G_1 , the next N_2 vertices are of G_2 , and so on so forth. Recall that W_+ , defined in section II-A, is the maximal set of the skeleton of G . and from the assumption of Theorem 1, we have $N_i > (n+1)$ for all $w_i \in W_+$. We first prove the result for the case when W_+ is a singleton, and then show how to lift this assumption.

Without loss of generality, we assume that $W_+ = \{w_1\}$. Let p_1 be the sub-configuration of p associated with G_1 . By assumption, p is contained in Q . So then, the sub-configuration p_1 is non-degenerate in \mathbb{R}^n , and hence, there must be $(n+1)$ vectors, say x_1, \dots, x_{n+1} , such that $p' := (x_1, \dots, x_{n+1})$ is a non-degenerate configuration in \mathbb{R}^n . Now define

$$L_{p'} := \{D(A_{ij})p \mid 1 \leq i, j \leq n+1, i \neq j\}.$$

From Lemma 14, \bar{G}_1 is a complete graph. Thus, $v_i \rightarrow v_j$ is an edge of \bar{G}_1 (and hence, of \bar{G}) for all $1 \leq i, j \leq n$. Thus, by Proposition 13, we have $L_{p'} \subseteq \mathcal{L}_p$. There are $n(n+1)$ vectors in $L_{p'}$, and from Lemma 19, the vectors in $L_{p'}$ are linearly independent.

The remaining $n \times (N - n - 1)$ vectors are constructed as follows. Since the configuration $p' = (x_1, \dots, x_{n+1})$ is non-degenerate in \mathbb{R}^n , for each x_i with $i \geq (n+2)$, we know from Proposition 11 that there are n vectors $\{x_{i_1}, \dots, x_{i_n}\}$ out of $\{x_1, \dots, x_{n+1}\}$ such that these n vectors, together with x_i form a non-degenerate configuration in \mathbb{R}^n . Now, with the choice of the n vectors $\{x_{i_1}, \dots, x_{i_n}\}$, we define

$$L_{x_i} := \{D(A_{ij})p \mid j = i_1, \dots, i_n\}.$$

Since w_1 is the maximal element, for any vertex w_k of the skeleton digraph H , there is a path from w_k to w_1 . Using the fact that G_i 's are strongly connected, we know that for any vertex v_j of G_1 , there is a path from v_i to v_j in G , and hence, $v_i \rightarrow v_j$ is an edge of \bar{G} . Then, using Proposition 13 again, we know that $L_{x_i} \subseteq \mathcal{L}_{p'}$.

We now show that the vectors in L_{x_i} are linearly independent. To see this, we define

$$X := (x^1, \dots, x^n) \in \mathbb{R}^{N \times n}.$$

Note that the vector $D(A_{ij})p$ is derived by concatenating the n vectors $A_{ij}x^1, \dots, A_{ij}x^n \in \mathbb{R}^N$. Thus, the n vectors in L_{x_i} are linearly independent if and only if the n matrices $\{A_{ii_1}X, \dots, A_{ii_n}X\}$ are linearly independent. By computation, the matrix $A_{ij}X$ satisfies the following condition: the i -th row of $A_{ij}X$ is $(x_j - x_i)^\top$ while all the other entries are zeros. Thus, it suffices to show that the n vectors $x_{i_1} - x_i, \dots, x_{i_n} - x_i$ are linearly independent. But, this follows from the fact that the configuration formed by the $(n+1)$ agents $x_{i_1}, \dots, x_{i_n}, x_i$ is non-degenerate in \mathbb{R}^n . We have thus proved that the n vectors in L_{x_i} are linearly independent.

The computation above furthermore shows the following fact: choose another $i' = (n+2), \dots, N$, and choose $x_{i'_1}, \dots, x_{i'_n}$ such that these n vectors, together with $x_{i'}$ form another non-degenerate configuration in \mathbb{R}^n . Construct $L_{x_{i'}}$ in the same way as L_{x_i} . Then, vectors in $L_{x_{i'}}$ are linearly independent of the vectors in L_{x_i} . Indeed, if $i \neq i'$, then the positions of the nonzero rows of $A_{ij}X$ and $A_{i'j'}X$ are different for any j and j' , and hence,

$$\text{tr}((A_{i'j'}X)^\top A_{ij}X) = (D(A_{ij})p)^\top D(A_{i'j'})p = 0$$

where $\text{tr}(\cdot)$ is the trace of a matrix. Using the same argument, we can show that the vectors in L_{x_i} are linearly independent of vectors in $L_{p'}$. Now define

$$L := L_{p'} \cup L_{x_{n+2}} \cup \dots \cup L_{x_N}.$$

Then, by construction, there are $n(n+1)$ vectors in $L_{p'}$, and n vectors in each L_{x_i} for $i = (n+2), \dots, N$. So then, there are

$$n(n+1) + \sum_{i=n+2}^N n = n \times N$$

vectors in $L \subset \mathcal{L}_p$, and they are linearly independent. Thus, we have

$$\dim \mathcal{L}_p = \dim P = n \times N,$$

and hence, \mathcal{L}_p satisfies the Lie algebra rank condition.

To conclude, we point out that the same analysis can be applied to the general case $|W_+| > 1$. Without loss of generality, we assume that $W_+ = \{1, \dots, k\}$ for some $k < q$. Let p_1, \dots, p_k be the sub-configurations of p associated with the subgraphs G_1, \dots, G_k of G , respectively. By assumption, p is contained in Q . So then, p_1, \dots, p_k are non-degenerate configurations in \mathbb{R}^n . Hence, for each $i = 1, \dots, k$, there is a non-degenerate sub-configuration p'_i of p_i in \mathbb{R}^n which is comprised of $(n+1)$ agents. Thus, we can construct $L_{p'_1}, \dots, L_{p'_k}$ in the same way as $L_{p'}$ in the previous case. Each $L_{p'_i}$ contains $n(n+1)$ linearly independent vectors. Furthermore, if $i \neq j$, then the positions of nonzero entries of vectors in $L_{p'_i}$ are different from those of vectors in $L_{p'_j}$. Thus, vectors in $L_{p'_i}$ are perpendicular to (and hence, independent of) vectors in $L_{p'_j}$.

Now consider an agent x_j which is not contained in p'_i for any $i = 1, \dots, k$. From the definition of W_+ and the fact that the G_i 's are strongly connected, we know that there exists at least a subgraph G_i , for $i = 1, \dots, k$, such that for any vertex v_{i_1} of G_i , there exists a path from v_j to v_{i_1} . So then, $v_j \rightarrow v_{i_1}$ is

an edge of \bar{G} for any vertex v_{i_l} of G_i . Thus, we can construct L_{x_j} as in the case when $W_+ = \{1\}$, but replace p' with p'_i . To the end, there are n vectors in L_{x_j} , and the vectors in L_{x_j} are perpendicular to vectors in $L_{x_{j'}}$ as long as $j \neq j'$. Let p' be the sub-configuration of p derived by taking the union of p'_i , for $i = 1, \dots, k$. Define

$$L := \left(\bigcup_{i=1}^k L_{p'_i} \right) \cup \left(\bigcup_{x_j \notin p'} L_{x_j} \right).$$

Then, there are $n \times N$ vectors in L , and they are linearly independent. This completes the proof. \square

Theorem 1 is then a consequence of Propositions 3 and 18, and the Rachevsky-Chow's Theorem. The path controllability is a consequence of a result of Sussmann and Liu [17].

Remark 1. We point out that the condition $p \in Q$ is also a necessary condition for \mathcal{L}_p to satisfy the Lie algebra rank condition, and thus a necessary condition for controllability. This follows from the fact that the operations of taking the Lie bracket of the control vector fields and of taking rotations of a configuration p commute. Now, if the configuration p is degenerate of rank k , we can always rotate it by Θ so that the last $n - k$ coordinates of each agent in $\Theta \cdot p$ are zero and thus the last $n - k$ entries of the corresponding vector fields for each agent are zero. Taking the Lie bracket of such vector fields always results in a vector field with the last $n - k$ coordinates for each agent being zero. They thus form an involutive Lie algebra of dimension kN and thus do not pass the Lie algebra rank condition.

IV. CONCLUSIONS

In this paper, we have investigated the controllability of a bilinear formation control model with underlying network topology described by a directed graph G . We have shown that the system is approximately path-controllable over the path-connected, open dense subset Q (defined in (2)) provided that G is weakly connected and each maximal component of G has more than $(n + 1)$ vertices. To establish the result, we have exhibited some relations between the transitive closure of G and the Lie algebra closure of a set of zero row-sum matrices that arose naturally in the study of the formation control model. Future work may focus on designing explicit control laws for steering the system to follow a specific path, and computing the least number of u_{ij} 's that are in need for the controllability of system (1).

REFERENCES

- [1] V. Gazi and K.M. Passino. A class of attractions/repulsion functions for stable swarm aggregations. *International Journal of Control*, 77(18):1567–1579, 2004.
- [2] L. Krick, M.E. Broucke, and B.A. Francis. Stabilisation of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82(3):423–439, 2009.
- [3] X. Chen. Gradient flows for organizing multi-agent system. In *American Control Conference (ACC)*, 2014, pages 5109–5114. IEEE, 2014.
- [4] X. Chen. Decentralized formation control with a quadratic Lyapunov function. In *American Control Conference (ACC)*, 2015. IEEE, 2015.

- [5] B.D.O. Anderson, C. Yu, S. Dasgupta, and A.S. Morse. Control of a three-coleader formation in the plane. *Systems & Control Letters*, 56(9):573–578, 2007.
- [6] M.-A. Belabbas, S. Mou, A.S. Morse, and B.D.O. Anderson. Robustness issues with undirected formations. In *Conference on Decision and Control (CDC)*, 2012, pages 1445–1450. IEEE, 2012.
- [7] Z. Sun, S. Mou, B.D.O. Anderson, and A.S. Morse. Formation movements in minimally rigid formation control with mismatched mutual distances. In *Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [8] U. Helmke, S. Mou, Z. Sun, and B.D.O. Anderson. Geometrical methods for mismatched formation control. In *The 53rd Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [9] S. Mou, A.S. Morse, M.-A. Belabbas, and B.D.O. Anderson. Undirected rigid formations are problematic. In *Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [10] M. Lorenzen and M.-A. Belabbas. Distributed local stabilization in formation control. In *Control Conference (ECC), 2014 European*, pages 2914–2919. IEEE, 2014.
- [11] M.-A. Belabbas. On global stability of planar formations. *Automatic Control, IEEE Transactions on*, 58(8):2148–2153, 2013.
- [12] X. Chen and R.W. Brockett. Centralized and decentralized formation control with controllable interaction laws. In *Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [13] X. Chen, M.-A. Belabbas, and T. Başar. Directed formation control with controllable interaction weights. *arXiv preprint arXiv:1412.6925*, submitted to 2015 *Conference on Decision and Control*, 2015.
- [14] R. Diestel. *Graph Theory*. Springer, 2010.
- [15] Z. Costello and M. Egerstedt. The degree of nonholonomy in distributed computations. In *Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [16] A.M. Bloch. *Nonholonomic mechanics and control*, volume 24. Springer Science & Business Media, 2003.
- [17] H.J. Sussmann and W. Liu. Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories. In *Conference on Decision and Control (CDC)*, 1991, pages 437–442. IEEE, 1991.

APPENDIX

We prove here Lemma 4 stated in Section II-C, and Lemma 14 stated in Section II-E

Proof of Lemma 4. Letting $p \in P^k$, we show that there is an open neighborhood $U \subset \mathbb{R}^{nN}$ of p , an open neighborhood $V \subset \mathbb{R}^{nN}$ of 0, and a diffeomorphism $f : U \rightarrow V$ such that

$$f(U \cap P^k) = V \cap \mathbb{R}^{d_k}$$

where \mathbb{R}^{d_k} is a linear subspace of \mathbb{R}^{nN} with the last $(nN - d_k)$ entries being zeros. For simplicity, but without loss of generality, we assume that

$$\text{rank}\{x_2 - x_1, \dots, x_{k+1} - x_1\} = k.$$

Denote by p_1 the sub-configuration formed by x_1, \dots, x_{k+1} , and by p_{-1} the sub-configuration formed by the remaining agents.

Choose an open neighborhood $U_1 \subset \mathbb{R}^{n(k+1)}$ of p_1 such that if $p'_1 = (x'_1, \dots, x'_{k+1}) \in U_1$, then

$$\text{rank}\{x'_2 - x'_1, \dots, x'_{k+1} - x'_1\} = k.$$

Choose any open neighborhood $U_{-1} \subset \mathbb{R}^{n(N-k-1)}$ of p_{-1} , and let

$$U := U_1 \times U_{-1}.$$

Then, U is an open neighborhood of p in \mathbb{R}^{nN} . For each $p' \in U$, define an n -by- k matrix as follows:

$$A_{p'_1} := (x'_2 - x'_1, \dots, x'_{k+1} - x'_1) \in \mathbb{R}^{n \times k}.$$

Choose an $n \times (n - k)$ matrix $B_{p'_1}$ such that $B_{p'_1}$ is of full-column rank, i.e., $\text{rank } B_{p'_1} = n - k$, and moreover, the columns of $B_{p'_1}$ are perpendicular to columns of $A_{p'_1}$, i.e., $B_{p'_1}^\top A_{p'_1} = 0$. Furthermore, we can choose $B_{p'_1}$ so that it depends smoothly on $p'_1 \in U_1$; indeed, we first find an $n \times (n - k)$ matrix B so that by shrinking U_1 if necessary, the matrix $(A_{p'_1}, B)$ is nonsingular for all $p'_1 \in U_1$. Then, by applying the Gram-Schmidt process, we get $B_{p'_1}$. Now for each $p'_1 \in U_1$, define an n -by- n matrix as follows:

$$L_{p'_1} := (A_{p'_1}, B_{p'_1})^\top \in \mathbb{R}^{n \times n}.$$

Then, by construction, $L_{p'_1}$ is invertible, depending smoothly on $p'_1 \in U_1$.

We now construct the diffeomorphism $f : U \rightarrow \mathbb{R}^{nN}$. First, for each $p' = (x'_1, \dots, x'_N) \in U$, we define a set of vectors $v_1(p'), \dots, v_N(p')$ in \mathbb{R}^n as follows:

$$v_i(p') := \begin{cases} x'_i - x_i & \text{if } i = 1, \dots, k+1, \\ L_{p'_1} x'_i - L_{p_1} x_i & \text{if } i = k+2, \dots, N. \end{cases} \quad (9)$$

Then, we define

$$f : p' \mapsto (v_1(p'), \dots, v_N(p')).$$

From (9), we know that the map f is smooth and open. Let V be the image of f , i.e., $V := f(U)$. Since $f(p) = 0$, we have that V is an open neighborhood of 0 in \mathbb{R}^{nN} .

We now show that the map $f : U \rightarrow V$ is invertible. Pick a vector $(v_1, \dots, v_N) \in V$ with $v_i \in \mathbb{R}^n$. Let

$$p'_1 := (v_1 + x_1, \dots, v_{k+1} + x_{k+1}).$$

Then, $p'_1 \in U_1$, and hence, $L_{p'_1}$ is invertible. Define

$$p'_{-1} := L_{p'_1}^{-1} (v_{k+2} + L_{p_1} x_{k+2}, \dots, v_N + L_{p_1} x_N),$$

Then, the map $f^{-1} : V \rightarrow U$ defined by

$$f^{-1} : (v_1, \dots, v_N) \mapsto (p'_1, p'_{-1})$$

is the inverse of f , and it is also smooth by the construction. Thus, f is a diffeomorphism between U and V .

We now show that the image of $U \cap P^k$ under f is $V \cap \mathbb{R}^{dk}$. First, note that if $p' \in U$, then $r_{p'} \geq k$ because

$$\text{rank} \{x'_2 - x'_1, \dots, x'_{k+1} - x'_1\} = k.$$

Moreover, the equality $r_{p'} = k$ holds if and only if each x'_i , for $i = k+2, \dots, N$, is in the column space of $A_{p'_1}$. Equivalently, $r_{p'} = k$ if and only if

$$B_{p'_1} x'_i = 0, \quad \forall i = k+2, \dots, N. \quad (10)$$

Then, following (9) and (10), we conclude that $r_{p'} = k$ if and only if the last $(n - k)$ entries of $v_i(p')$ are zero for all $i = k+2, \dots, N$. The total number of these zero entries are $(n - k)(N - k - 1)$. On the other hand, all the other entries of $v_i(p')$'s are free to choose as long as $(v_1(p'), \dots, v_N(p'))$ is in V . The number of these free entries is then

$$nN - (n - k)(N - k - 1)$$

which is equal to dk . Thus, we conclude that

$$f(U \cap P^k) = V \cap \mathbb{R}^{dk}.$$

This completes the proof. \square

We now prove Lemma 14.

Proof of Lemma 14. First, note that if G is strongly connected, then its transitive closure \bar{G} is a complete graph. Indeed, for any pair of vertices (v_i, v_j) , there is a path from v_i to v_j , and also a path from v_j to v_i in G . Thus, each \bar{G}_i , for $i = 1, \dots, q$, is a complete graph.

Next, we show that $\bar{G}_1, \dots, \bar{G}_q$ form the coarse strong component decomposition of \bar{G} . Let V_i be the vertex set of G_i , and of \bar{G}_i as well. Choose any subset $\{i_1, \dots, i_m\}$ of $\{1, \dots, q\}$. Let

$$V' := \sqcup_{j=1}^m V_{i_j},$$

and let G' be the subgraph of \bar{G} induced by V' . It suffices to show that G' is not strongly connected since otherwise, we would have a strong component decomposition with the number of components strictly less than q . The proof is done by contradiction. Suppose that G' is strongly connected; then for any vertex $v_{i'_k} \in V_{i'_k}$ and any vertex $v_{i'_l} \in V_{i'_l}$, for $k \neq l$, there is a path from $v_{i'_k}$ to $v_{i'_l}$ in G' (and hence, in \bar{G}). So then, by the definition of transitive closure (Def. 7), there is a path from $v_{i'_k}$ to $v_{i'_l}$ in G . Thus, we have that $w_{i_l} \succ w_{i_k}$. But conversely, we can apply the same argument, and have that $w_{i_k} \succ w_{i_l}$ which is a contradiction. So then, the subgraph G' can not be strongly connected. Thus, we have shown that $\bar{G}_1, \dots, \bar{G}_q$ form the coarse strong component decomposition of \bar{G} .

It now remains to show that \bar{H} is the skeleton digraph of \bar{G} . First, we show that \bar{H} is acyclic. Suppose not, then there is a cycle

$$w_{i_1} \rightarrow w_{i_2} \rightarrow \dots \rightarrow w_{i_k} \rightarrow w_{i_1}$$

contained in \bar{H} . Then again, by the definition of transitive closure, there is a path from w_{i_l} to $w_{i_{l+1}}$, for all $l = 1, \dots, k - 1$, and a path from w_{i_k} to w_{i_1} . Using these paths, we obtain a cycle in H which contradicts the fact that H is acyclic. Thus, the digraph \bar{H} is acyclic. It now suffices to show that if there is a path from w_i to w_j in H , then $w_i \rightarrow w_j$ is an edge of the digraph \bar{H} . Let

$$w_{i_1} \rightarrow w_{i_2} \rightarrow \dots \rightarrow w_{i_k}$$

be the path from w_i to w_j with $i_1 = i$ and $i_k = j$. Pick a vertex $v_{i_l} \in V_{i_l}$ for each $l = 1, \dots, k$. Since $w_{i_l} \rightarrow w_{i_{l+1}}$ is an edge of H , from the definition of the skeleton digraph, there is a path from a vertex of G_{i_l} to a vertex of $G_{i_{l+1}}$. But since G_{i_l} and $G_{i_{l+1}}$ are strongly connected, there is a path from v_{i_l} to $v_{i_{l+1}}$ in G , and this holds for each $l = 1, \dots, k - 1$. Using these paths, we then obtain a path from v_{i_1} to v_{i_k} in G , and hence, $v_{i_1} \rightarrow v_{i_k}$ is an edge of \bar{G} . Since $v_{i_1} \in V_i$ and $v_{i_k} \in V_j$, we conclude that $w_i \rightarrow w_j$ is an edge of \bar{H} . This completes the proof. \square